

On a problem of resonance with exponential nonlinearity

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Abstract

Consider the following semilinear elliptic problem on $B = \{x \in \mathbb{R}^2 : |x| < 1\}$

$$\begin{aligned} -\Delta u &= \lambda_1 u + e^u + f, & \text{in } B \\ u &= 0 & \text{on } \partial B \end{aligned}$$

with f satisfying the following condition : $f \in L^r(\Omega)$, for some $r > 2$ and

$$\int_B f \phi_1 < 0.$$

Where ϕ_1 is the eigen function of $(-\Delta)$ corresponding to the first eigenvalue λ_1 in $H_0^1(B)$. We shall find the existence of a radial solution of this PDE. We shall use degree theory to get the existence starting from a suitable with known solution with its degree. Connecting those two PDE's by homotopy and getting the uniform estimate for the connecting PDE's we shall achieve our result.

1 Introduction

Existence of solutions for semilinear elliptic Dirichlet problems

$$\begin{aligned} -\Delta u &= g(x, u), & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

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with distinct behaviour of

$$\frac{g(x, s)}{s} \text{ as } s \rightarrow \pm\infty$$

is difficult to establish in the case when

- I. $g(x, 0) \neq 0$ (so there is no trivial solution).
- II. There is resonance in one direction, and
- III. The problem is superlinear in the other.

The problem seems to be particularly harder to deal if such a resonance is at the first eigenvalue of the laplacian, in view of the fact that the corresponding first eigenfunction has a definite sign. It is a problem of this kind that we will treat here, namely

$$\begin{aligned} -\Delta u &= \lambda_1 u + e^u + f, & \text{in } B \\ u &= 0 & \text{on } \partial B \end{aligned} \tag{1.2}$$

where $B = \{x \in \mathbb{R}^2 : |x| < 1\}$ with f is a radial function such that, $f \in L^r(\Omega)$, for some $r > 2$ and ϕ_1 is the eigen function of $(-\Delta)$ corresponding to the first eigenvalue λ_1 in $H_0^1(B)$. Solving 1.2 is particularly hard since e^u is in some sense the critical nonlinearity (see[1]). Also see [3] where the restriction on the exponent ' p ' is essentially due to lack of a priori bounds. In some sense, specially in the context of the approach we have adapted the result seems optimal (see [4]), which seems to indicate that the bounds are hard to establish.

Let us assume that $\phi > 0$ in B and

$$\int_B \phi_1^2 = 1, \text{ and } \int_B |\nabla \phi_1|^2 = \lambda_1 \tag{1.3}$$

Multiplying the equation 1.2 by ϕ_1 and integration by parts we get

$$\int_B e^u \phi_1 + \int_B f \phi_1 = 0, \text{ i.e. } \int_B f \phi_1 = - \int_B e^u \phi_1 < 0. \tag{1.4}$$

The above condition is necessary for the existence of the solution. We shall try to find out whether this condition is sufficient for the existence of the solution in the context of radial solution and shall prove the following theorem.

Theorem 1.1. *If*

$$0 < - \int_B f \phi_1 < 4\pi, \tag{1.5}$$

then the equation(1.2) has a nontrivial radial solution.

In some sense the integral value 4π in (1.5) may be optimal. In [2] the authors have given some example (Section 6, proposition 2) where they have shown the breaking of symmetry as the integral value in (1.5) goes higher, infact it peaks up more and more concentrating points as the integral value increases. Our proof uses the well known degree and homotopy arguments. The required bounds for the homotopy, established using the results of Brezis, Marle in [1]

2 The comparison equation

Consider the equation

$$\begin{aligned} -\Delta u &= \lambda_1 u + g(u), & \text{in } B \\ u &= 0 & \text{on } \partial B \end{aligned} \quad (2.1)$$

where

$$g(t) = \begin{cases} \sin t & \text{if } -\pi \leq t \leq \pi \\ 0 & \text{otherwise} \end{cases} \quad (2.2)$$

Theorem 2.1. *If $\lambda_2 - \lambda_1 > 1$ then the equation (2.1) has only 0 solution, and the solution is non-degenerate and the 'L-S Degree' is -1 .*

Proof. First note that any non-zero H_0^1 solution of the equation (2.1) has to change sign. Multiplying (2.1) by ϕ_1 and integrating by parts we get

$$\int_B g(u) \phi_1 = 0.$$

hence u has to change sign. Let $u \neq 0$ be a solution of (2.1). Defining $g(u)/u = 1$ at $u = 0$ we can re-write the equation (2.1) as

$$-\Delta u = \left[\lambda_1 + \frac{g(u)}{u} \right] u. \quad (2.3)$$

Note that $g(u)/u \leq 1$ for all $x \in B_1$ and < 1 on a positively measured subset of B_1 , (as u is non-zero solution). Consider the following eigenvalue problems

$$-\Delta u = \mu \left[\lambda_1 + \frac{g(u)}{u} \right] u, \text{ in } B, \quad u = 0 \text{ on } \partial\Omega \quad (2.4)$$

$$-\Delta u = \mu \lambda_2 u, \text{ in } B, \quad u = 0 \text{ in } \partial\Omega \quad (2.5)$$

Note that $\lambda_1 + \frac{g(u)}{u} \leq \lambda_2$ and the strict inequality holds on a +vely measured set. Hence we get

$$\mu_k \left[\lambda_1 + \frac{g(u)}{u} \right] > \mu_k(\lambda_2), \forall k. \quad (2.6)$$

Now u being a sign changing solution and $\mu_2(\lambda_2) = 1$ implies that $\mu_k \left[\lambda_1 + \frac{g(u)}{u} \right] > 1, \forall k \geq 2$. Now u being a sign-changing solution of (2.4) we have $\mu_k \left[\lambda_1 + \frac{g(u)}{u} \right] = 1$, which is contradictory. Hence 0 is only solution of (2.1).

The linearized equation of (2.4) at 0 is

$$-\Delta v = (\lambda_1 + 1)v, \text{ in } B, v = 0 \text{ in } \partial\Omega. \quad (2.7)$$

Now $\lambda_2 > \lambda_1 + 1$ implies 0 is the only solution of (2.7) and hence 0 is the non-degenerate solution of (2.1). Also note that 0 being only solution of the equation, the degree of the solution is -1 .

□

Remark. In the context of our theorem, there is no loss of generality by assuming $\lambda_2 - \lambda_1 > 1$, since we can always replace $g(u)$ by $\varepsilon g(u)$ for the comparison equation.

Now consider the equation

$$\begin{aligned} -\Delta u_t &= \lambda_1 u_t + t(e^{u_t} + f) + (1-t)g(u_t), & \text{in } B \\ u_t &= 0 & \text{on } \partial B \end{aligned} \quad (2.8)$$

We shall show that the solutions u_t is bounded uniformly in $L^\infty(B)$.

Lemma 2.2. *Let R_t be a critical point of u_t such that $\exists t_0 \in [0, 1]$ with $\lim_{t \rightarrow t_0} u_t(R_t) \rightarrow \infty$. Then $R_t \rightarrow 0$ as $t \rightarrow t_0$.*

Proof. u_t is radial. Hence from the equation (2.8) we have

$$-(ru_t')' = \lambda_1 r u_t + t(e^{u_t} + f)r + (1-t)g(u_t)r.$$

The first eigenfunction ϕ_1 of Δ is also radial. So multiplying the above by ϕ_1 and integrating it by parts over $[R_t, 1]$ we get

$$-\int_{R_t}^1 \phi_1 dr = \int_{R_t}^1 \lambda_1 r u_t \phi_1 dr + \int_{R_t}^1 [t(e^{u_t} + f) + (1-t)g(u_t)] \phi_1 r dr$$

Now we have

$$-\int_{R_t}^1 \phi_1 dr = \int_{R_t}^1 \lambda_1 r u_t \phi_1 dr + R_t u_t(R_t) |\phi_1'(R_t)|.$$

Hence

$$R_t u_t(R_t) |\phi_1'(R_t)| = \int_{R_t}^1 t(e^{u_t} + f) \phi_1 r dr + \int_{R_t}^1 (1-t)g(u_t) \phi_1 r dr$$

$$\begin{aligned}
&\leq \frac{1}{2\pi} \int_A t(e^{u_t} + f)\phi_1 dx + C \\
&= \frac{1}{2\pi} \int_A te^{u_t}\phi_1 dx + \frac{1}{2\pi} \int_A tf\phi_1 dx + C \\
&\leq \frac{1}{2\pi} \int_B t(e^{u_t} + f)\phi_1 dx + \frac{1}{2\pi} \int_{B \setminus A} t|f|\phi_1 dx + C \\
&\leq \frac{1}{2\pi} \int_B t(e^{u_t} + f)\phi_1 dx + C \\
&\leq \frac{1}{2\pi} \int_B (1-t)|g(u_t)|\phi_1 dx + C \\
&\leq C
\end{aligned}$$

Now if $R_t \rightarrow 0$ as $t \rightarrow t_0$, we have $|\phi'_1(R_t)| > 0, \forall t \in (t_0 - \varepsilon, t_0 + \varepsilon) \cap I$. So we have

$$u_t(R_t) \leq \frac{C}{R_t |\phi'_1(R_t)|} \leq C_1, \forall t \in (t_0 - \varepsilon, t_0 + \varepsilon) \cap I.$$

Which is contradictory. And hence we have the result. \square

Let us write u_t as

$$u_t = T_t \phi_1 + \omega_t. \quad (2.9)$$

Then ω_t satisfies

$$\begin{aligned}
-\Delta \omega_n &= \lambda_1 \omega_n + t_n(e^{u_n} + f) + (1 - t_n)g(u_{t_n}), & \text{in } B \\
\omega_n &= 0 & \text{on } \partial B
\end{aligned} \quad (2.10)$$

Lemma 2.3. ω_t is uniformly $L^2(B)$ bounded. i.e.

$$\sup_{t \in [0,1]} \|\omega_t\|_{L^2(B)} < \infty \quad (2.11)$$

Proof. If possible let us assume there is a sequence t_n such that $\|\omega_{t_n}\|_{L^2(B)} \rightarrow \infty$. Let us denote $\omega_n = \omega_{t_n}$. Let H_n satisfies

$$\begin{aligned}
-\Delta H_n &= \lambda_1 \omega_n + t_n f + (1 - t_n)g(u_{t_n}), & \text{in } B \\
H_n &= 0 & \text{on } \partial B
\end{aligned} \quad (2.12)$$

Divide 2.12 by $\|\omega_n\|_2$ and we have

$$\begin{aligned}
-\Delta \left(\frac{H_n}{\|\omega_n\|_2} \right) &= \lambda_1 \frac{\omega_n}{\|\omega_n\|_2} + t_n \frac{f}{\|\omega_n\|_2} + (1 - t_n) \frac{g(u_{t_n})}{\|\omega_n\|_2}, & \text{in } B \\
\frac{H_n}{\|\omega_n\|_2} &= 0 & \text{on } \partial B
\end{aligned} \quad (2.13)$$

Note that $\left\| \frac{\omega_n}{\|\omega_n\|_2} \right\| = 1$ and

$$\lim_{n \rightarrow \infty} t_n \left\| \frac{f_n}{\|\omega_n\|_2} \right\|_2 = \lim_{n \rightarrow \infty} (1 - t_n) \left\| \frac{g(u_{t_n})}{\|\omega_n\|_2} \right\|_2 = 0$$

Hence by regularity we have $\frac{H_n}{\|\omega_n\|_2} \in H^2 \cap H_0^1$ $\left\| \frac{H_n}{\|\omega_n\|_2} \right\|_{H^2} < C, \forall n$, and hence we have $\frac{H_n}{\|\omega_n\|_2} \rightharpoonup L$ in $H_0^1(B)$. Also note that $\left\| \frac{\omega_n}{\|\omega_n\|_2} \right\|_2 = 1$ implies there is $N \in L^2(B)$ such that $\frac{\omega_n}{\|\omega_n\|_2} \rightharpoonup N$ in $L^2(B)$ and we have

$$\begin{aligned} -\Delta L &= \lambda_1 N, & \text{in } B \\ L &= 0 & \text{on } \partial B \end{aligned} \quad (2.14)$$

Multiplying (2.14) by ϕ_1 and integrating by parts we get

$$\int_B (L - N) \phi_1 = 0. \quad (2.15)$$

Now note that

$$\begin{aligned} -\Delta(\omega_n - H_n) &= t_n e^{u_{t_n}} > 0, & \text{in } B \\ \omega_n - H_n &= 0 & \text{on } \partial B \end{aligned} \quad (2.16)$$

Hence by maximum principle we have $\omega_n - H_n \geq 0$ and hence

$$\begin{aligned} \int_B \left(\frac{\omega_n}{\|\omega_n\|_2} - \frac{H_n}{\|\omega_n\|_2} \right) \phi_1 &\geq 0 \\ \lim_{n \rightarrow \infty} \int_B \left(\frac{\omega_n}{\|\omega_n\|_2} - \frac{H_n}{\|\omega_n\|_2} \right) \phi_1 &\geq 0 \\ \int_B (N - L) \phi_1 &\geq 0 \end{aligned} \quad (2.17)$$

Hence from (2.15) and (2.17) we have $N = L$. So L satisfies

$$\begin{aligned} -\Delta L &= \lambda_1 L, & \text{in } B \\ L &= 0 & \text{on } \partial B \end{aligned} \quad (2.18)$$

And we get $L = l\phi_1$ for some $l \in \mathbb{R}$ and we have $\frac{\omega_n}{\|\omega_n\|_2} \rightharpoonup l\phi_1$ and hence $l = 0$, also we have

$$\frac{\omega_n}{\|\omega_n\|_2} \rightharpoonup 0 \text{ in } L^2(B), \text{ and } \frac{H_n}{\|\omega_n\|_2} \rightharpoonup 0 \text{ in } H^2(B)$$

Then by compact embedding we get $\frac{H_n}{\|\omega_n\|_2} \rightarrow 0$ in $H_0^1(B)$ and hence $\frac{H_n}{\|\omega_n\|_2} \rightarrow 0$ in $C^0(\overline{B}) \cap C^1(B)$. So we have

$$\left\| \frac{H_n}{\|\omega_n\|_2} \right\| \rightarrow 0, \quad (2.19)$$

for some positive constant C .

Note that ϕ_1 , the first positive eigenfunction lies in the interior of the cone of positive functions in the space $C_0^1(\overline{B})$. So we have

$$\frac{H_n}{\|\omega_n\|_2} \leq C\phi_1 \quad (2.20)$$

for some positive constant C_1 . Now H_n satisfies

$$-\Delta H_n = \lambda_1 \omega_n + t_n f + (1 - t_n)g(u_n).$$

Multiplying both sides of the above by ω_n and integrating by parts we get

$$\begin{aligned} & \int_B H_n \left(\lambda_1 \omega_n + t_n (e^{u_n} + f) + (1 - t_n)g(u_n) \right) dx \\ &= \lambda_1 \|\omega_n\|^2 + \int_B \left(t_n f \omega_n + (1 - t_n)g(u_n) \omega_n \right) dx \\ \text{i.e. } & \int_B \lambda_1 \frac{H_n}{\|\omega_n\|_2} \frac{\omega_n}{\|\omega_n\|_2} + \frac{1}{\|\omega_n\|_2^2} \int_B \left(t_n H_n e^{u_n} + t_n H_n f + (1 - t_n)H_n g(u_n) \right) dx \\ &= \lambda_1 + \int_B t_n \frac{f}{\|\omega_n\|_2} \frac{\omega_n}{\|\omega_n\|_2} dx + (1 - t_n) \int_B \frac{g(u_n)}{\|\omega_n\|_2} \frac{\omega_n}{\|\omega_n\|_2} dx \end{aligned} \quad (2.21)$$

The LHS. of (2.21) can be represented as

$$\begin{aligned} & \int_B \lambda_1 \frac{H_n}{\|\omega_n\|_2} \frac{\omega_n}{\|\omega_n\|_2} + \frac{t_n}{\|\omega_n\|_2} \int_B e^{u_n} \left(\frac{H_n}{\|\omega_n\|_2} - C_1 \phi_1 \right) + t_n \int_B \frac{H_n}{\|\omega_n\|_2} \frac{f}{\|\omega_n\|_2} \\ &+ (1 - t_n) \int_B \frac{g(u_n)}{\|\omega_n\|_2} \frac{H_n}{\|\omega_n\|_2} dx + C_1 t_n \int_B e^{u_n} \frac{\phi_1}{\|\omega_n\|_2} dx \end{aligned}$$

Now multiplying (2.8) by ϕ_1 and integrating by parts we get

$$-t_n \int_B e^{u_n} \phi_1 = t_n \int_B f \phi_1 dx + (1 - t_n) \int_B g(u_n) \phi_1 dx. \quad (2.22)$$

So we have $|t_n \int_B e^{u_n} \phi_1| < \infty$. Hence

$$\lim_{n \rightarrow \infty} t_n \int_B \frac{e^{u_n} \phi_1}{\|\omega_n\|_2} \rightarrow 0. \quad (2.23)$$

Then from (2.20) we have as $n \rightarrow \infty$, LHS. ≤ 0 . Similarly we can show that as $n \rightarrow \infty$ RHS. $\rightarrow \lambda_1 > 0$ which is contradictory. Hence we have $\|\omega_t\|_2$ bounded uniformly. \square

Lemma 2.4. T_t is bounded.

Proof. We have taken $u_n = T_n\phi_1 + \omega_n$ and $\|\omega_n\|_2 < \infty$. Let $\tilde{\omega}_n$ satisfies

$$\begin{aligned} -\Delta\tilde{\omega}_n &= \lambda_1\omega_n + t_nf + (1-t_n)g(u_n), & \text{in } B \\ \tilde{\omega}_n &= 0 & \text{on } \partial B \end{aligned} \quad (2.24)$$

Note that the RHS. of the equation is uniformly bounded in $L^2(B)$ and hence by regularity theory we have $\tilde{\omega}_n \in H^2(B) \cap H_0^1(B)$. And hence by Sobolev embedding theorem $\tilde{\omega}_n \in C^1(B) \cap C^0(\bar{B})$. Now

$$\begin{aligned} -\Delta(\omega_n - \tilde{\omega}_n) &= t_ne^{u_n}, & \text{in } B \\ \omega_n - \tilde{\omega}_n &= 0 & \text{on } \partial B \end{aligned} \quad (2.25)$$

By maximum principle we have $\omega_n - \tilde{\omega}_n > 0$ in B . So ω_n is bounded from below uniformly on n .

If possible let us suppose that there is t_n such that $T_n := T_{t_n} \rightarrow \infty$. Let us first show that $t_n \rightarrow 0$ as $n \rightarrow \infty$. If not, let up to a subsequence $t_n \rightarrow t_0 \neq 0$. So for large n we have from (2.22)

$$\int_B \phi_1(e^{u_n} + f) + \frac{1-t_n}{t_n} \int_B g(u_n)\phi_1 = 0.$$

Note that in any compact set $K \subset B$, $\phi_1 e^{u_n} \rightarrow \infty$ uniformly as ω_n bounded below. So the above inequality can't hold as all other terms are bounded. Hence $\lim_{n \rightarrow \infty} t_n \rightarrow 0$.

Now let us show that for n large $u_n \geq 0$. Divide $[0, 1]$ into two fixed intervals $[0, 1 - \delta]$ and $(1 - \delta, 1]$, for some small positive number δ . Note that there is N_1 such that $u_n \geq 0$ in $[0, 1 - \delta]$ for all $n \geq N_1$. Form (2.25) and using Hopf maximum principle we have

$$\frac{\partial \tilde{\omega}_n}{\partial \eta} \geq \frac{\partial \omega_n}{\partial \eta}.$$

Using elliptic regularity and Sobolev embedding we have $\frac{\partial \tilde{\omega}_n}{\partial \eta}$ is bounded uniformly on ∂A . And hence $\frac{\partial \omega_n}{\partial \eta}$ is bounded uniformly on ∂A . Note that $\phi_1'(1) < 0$, implies there is N large such that

$$u_n'(1) = T_n\phi_1'(1) - \omega_n'(1) < 0. \quad (2.26)$$

Hence u_n is positive near the boundary for $n \geq N$. Hence u_n is positive near the boundary for n large. Now let u_n changes sign. Define

$$a_n = \sup\{r \in (0, 1) : u_n(a_n) = 0\}.$$

Clearly $u_n'(a_n) \geq 0$. First note that $\lim_{n \rightarrow \infty} a_n \rightarrow 1$, if not let up to a subsequence $a_n \rightarrow a < 1$. Now $u_n(a_n) = T_n\phi_1(a_n) + \omega_n(a_n) \geq T_n\phi_1(a + 1/n) + \omega_n(a_n) \rightarrow \infty$ as $n \rightarrow \infty$, which is a contradiction. Now note that

$$u_n'(r) = T_n\phi_n'(r) + \omega_n'(r) \leq T_n\phi_n'(1 - \delta) + \omega_n'(r) \rightarrow -\infty, \text{ as } n \rightarrow \infty,$$

for all $r \in (1 - \delta, 1)$. Now $a_n \rightarrow 1$ and $u'_n(a_n) \geq 0$ contradicts the above. Hence $u_n \geq 0$ for n large.

Now $u_n \geq 0$ implies $g(u_n) \geq 0$ also f being bounded $e^{u_n} + f \geq 0$ and strictly on a positive measured set, and we have

$$t_n \int_B e^{u_n} \phi_1 + t_n \int_B f \phi_1 dx + (1 - t_n) \int g(u_n) dx > 0,$$

contradicting (2.22). Hence $T_n \rightarrow \infty$.

If possible let $T_n \rightarrow -\infty$. Let us write $u_n = -T_n \phi_1 + \omega_n$. Then note that $T_n \rightarrow \infty$. First note that

$$\int_A \omega_n^+ \phi_1 = \int_A \omega_n^- \phi_1 < \infty.$$

Hence $\lim_{n \rightarrow \infty} \mu\{x : \omega_n^+(x) > n\} = 0$. Then we have as $T_n \rightarrow \infty$, $\lim_{n \rightarrow \infty} \mu\{x : -T_n \phi_1 + \omega_n(x) > -\pi\} = 0$. And thus $\lim_{n \rightarrow \infty} \int_A g(u_n) \phi_1 = 0$, as g and ϕ_1 are both bounded. Then from (2.22) we have either $\int_B e^{u_n} \phi_1 \rightarrow -\int_B f \phi_1$ or $t_n \rightarrow 0$ and in this case $\lim_{n \rightarrow \infty} \int_A t_n e^{u_n} \phi_1 = 0$.

Case I : Let us first assume $t_n \rightarrow 0$. Define $v_n := \omega_n - \tilde{\omega}_n$. Then from (2.25) we have

$$\begin{aligned} -\Delta v_n &= V_n e^{v_n}, & \text{in } B \\ v_n &= 0 & \text{on } \partial B \end{aligned} \tag{2.27}$$

Where $V_n = t_n e^{-T_n \phi_1 + \tilde{\omega}_n}$. Note that $\|\tilde{\omega}_n\|_\infty < \infty$ uniformly on n . And hence we have $\|V_n\|_\infty < \infty$, uniformly on n . Now $V_n e^{v_n} = t_n e^{u_n}$.

Let us first assume that, there is $0 < a < 1$ such that $\omega_n < 0$ on $(a, 1)$ for all n . Then from (2.22), we have

$$\int_{B(0,a)} t_n e^{u_n} \leq \frac{1}{\phi_1(a)} [-t_n \int_B f \phi_1 + (1 - t_n) \int_B g(u_n) \phi_1].$$

So $\lim_{n \rightarrow \infty} \int_{B(0,a)} t_n e^{u_n} = 0$. Now as in $(a, 1)$, $\omega_n < 0$, by choosing a properly and n large we have

$$\int_B V_n e^{v_n} = \int_B t_n e^{u_n} < 4\pi. \tag{2.28}$$

So by the result of Brezis Merle [1] we have $\|\omega_n - \tilde{\omega}_n\|_\infty < C$, for some positive constant C , for all n . And hence we have $\|\omega_n\| < C$, for all n . Then using regularity from (2.10) we get $\|\omega_n\|_{C^1(\bar{B})} < C$, for all n . Now ϕ_1 being in the interior of the cone of positive functions in $C^1(\bar{B})$ we have

$$u_n = -T_n \left(\phi_1 - \frac{\omega_n}{T_n} \right) < 0, \text{ for } n \text{ large.}$$

Now choose an interval $(a, 1)$ such that

$$\int_a^1 \phi_1 dr + 1/4 \int_a^1 f \phi_1 dr < 0. \quad (2.29)$$

also in the interval $[0, a]$, $u_n \downarrow 0$ uniformly. Hence for n large,

$$\int_0^a e^{u_n} \phi_1 dr + 1/4 \int_a^1 f \phi_1 dr < 0. \quad (2.30)$$

Combining (2.29) and (2.30) we get

$$\int_0^1 e^{u_n} \phi_1 dr + \int_a^1 f \phi_1 dr < 0$$

as $\int_0^1 f \phi_1 < 0$. Now note that $g(u_n) \leq 0$ for n large, hence $\int_0^1 g(u_n) \leq 0$. So we have

$$t_n \int_0^1 (e^{u_n} + f) \phi_1 dr + (1 - t_n) \int_0^1 g(u_n) dr < 0.$$

contradicting (2.22). So ω_n has to positive value in $(a, 1)$ for n large.

Now we shall show that in $(a, 1)$, $\|\omega_n\|_{L^\infty(a,1)} < C$ for some positive constant C for all n large. If not we have for any $M_n \rightarrow \infty$ there is $\delta_n > 0$ such that $\mu\{x : \omega_n(x) > M_n\} \geq \delta_n$ and there is $p_n \in (a, 1)$ such that $\omega_n(p_n) > M_n$. Let $p_n \rightarrow p$ as $n \rightarrow \infty$ (up to a subsequence). Take $0 < p' < \inf\{p, a\}$. claim that in (p', a) , $\omega_n \geq M_n$ for all n . If not, we shall find q_n such that $\omega_n(q_n) < M_n$, $\forall n$. Hence there is a point of maxima R_n of ω_n in (p', a) with $\lim_{n \rightarrow \infty} \omega_n(R_n) \rightarrow \infty$ and $R_n \rightarrow 0$. Now similarly as proved in lemma[2.2] we can show the same result for ω_n . Which gives us a contradiction.

Now choose M large enough so that $\phi(a)M(p - p') > \int_{B_1(0)} \omega_n^- \phi_1$. Then we have for n large $M_n > M$ and

$$\int_{B_a(0)} \omega_n^+ \phi_1 \geq \phi(a) \int_{p'}^a M dx = \phi(a)M(p - p') > \int_{B_1(0)} \omega_n^- \phi_1.$$

Which is contradictory. So in $(a, 1]$, $\|\omega_n\|_{L^\infty(a,1)} < C$.

Now from 2.22 we have

$$\int_{B_a(0)} t_n e^{u_n} \leq \frac{1}{\phi_1(a)} \left[t_n \int_B |f| \phi_1 dx + (1 - t_n) \int_B \phi_1 dx \right] < \infty$$

and

$$\int_{B \setminus B_a(0)} t_n e^{u_n} = \int_{B \setminus B_a(0)} t_n e^{-T_n \phi_1} e^C < C.$$

So using theorem[1.1] we conclude that $\omega_n - \tilde{\omega}_n \in L^\infty(B)$, and $\|\omega_n - \tilde{\omega}_n\|_{L^\infty(B)} \leq C$, and thus we have $\|\omega_n\|_{L^\infty(B)} < C$. Using regularity we have for n large $\omega_n \in C_0^1(\overline{B})$. Now using the same cone condition we have $-T_n\phi_1 + \omega_m < 0$ and $-T_n\phi_1 + \omega_m \rightarrow -\infty$ in any compact subset of B . Hence $(1-t_n) \int_B g(u_n)\phi_1 \leq 0$ for n large and $\int_K e^{u_n} \rightarrow 0$ as $n \rightarrow \infty$ for any compact $K \subset B$. Using the fact $\int_B f\phi_1 < 0$ we have

$$t_n \int_B (e^{u_n} + f)\phi_1 dx + (1-t_n) \int_B g(u_n)\phi_1 dx < 0.$$

Which is contradictory. Hence $T_n \rightarrow \infty$.

Case II : Now let $\int_B e^{u_n}\phi_1 \rightarrow -\int_B f\phi_1$. We have $-\int_B f\phi_1 < 4\pi$. So for n large $t_n \int_B e^{u_n}\phi_1 < 4\pi$. Now for any $0 < a < 1$ we have shown that $\|\omega_n\|_{L^\infty(a,1)} < C$. As $T_n \rightarrow -\infty$, we have $\left\|\frac{\omega_n}{T_n}\right\|_{L^\infty(a,1)} \rightarrow 0$ as $n \rightarrow \infty$. Hence $\int_{B \setminus B_a(0)} t_n e^{u_n} \rightarrow 0$ as $n \rightarrow \infty$. Now choose δ such that $-\int_B f\phi_1 < 4\pi(1-\delta)$. Choose a such that $\phi(a) = 1-\delta$. Then

$$t_n \int_{B_a(0)} e^{u_n} \leq \frac{t_n}{\phi_1(a)} \int_{B_a(0)} e^{u_n} \phi_1 dx < 4\pi.$$

Combining both the integrals we have $t_n \int_B e^{u_n} < 4\pi$ for n large. And similarly as above we can arrive at the same contradiction.

Hence in both the cases we have $T_n < C$ for some positive constant C .

□

Theorem 2.5. $\|u_t\|_{L^\infty(B)} < C$, for some positive constant C .

Proof. If not then there exists a sequence t_n and a sequence of solutions u_n such that $\|u_n\|_{L^\infty(B)} \rightarrow \infty$ as $n \rightarrow \infty$. Now expressing $u_n = t_n\phi_1 + \omega_n$, we have shown that T_n is bounded. Hence $\|\omega_n\|_{L^\infty(B)} \rightarrow \infty$ as $n \rightarrow \infty$. Also we know that $R_n \rightarrow 0$ where $\omega_n(R_n) \rightarrow \infty$ and $\omega'_n(R_n) = 0$. And as before we can show that for any sequence q_n with $\omega_n(q_n) \rightarrow \infty$, $q_n \rightarrow 0$. Hence for any $1 > \delta > 0$

$$\|u_n\|_{L^\infty(B \setminus B_\delta(0))} < C_\delta, \quad (2.31)$$

for some $C_\delta > 0$

Now we shall establish a contradiction for two different cases.

Case I : Let $t_n \rightarrow t_0 \neq 0$. Let $v_n = \omega_n - \tilde{\omega}_n$. Then v_n satisfies 2.27. From 2.22 we have

$$\lim_{n \rightarrow \infty} \int_B e^{u_n} \phi_1 = - \int_B f \phi_1 + \frac{1-t_0}{t_0} \lim_{n \rightarrow \infty} \int_B g(u_n) \phi_1. \quad (2.32)$$

Hence there is a constant C such that $\int_B e^{u_n} \phi_1 < C$ uniformly for large n . Also for any $0 < a < 1$ we have shown that ω_n is uniformly bounded in $B \setminus B_a(0)$. Hence

$$\int_{B \setminus B_a(0)} e^{v_n} = \int_{B \setminus B_a(0)} e^{\omega_n - \tilde{\omega}_n} < C,$$

as $\tilde{\omega}_n$ bounded uniformly in B . Now

$$\int_{B_a(0)} e^{v_n} = \int_{B_a(0)} e^{u_n} e^{-(T_n \phi_1 + \tilde{\omega}_n)} \leq \frac{C}{\phi_1(a)} \int_B e^{u_n} \phi_1 \leq C. \quad (2.33)$$

Combining (2.32), (2.33) we have $\|v_n\|_{L^1(B)} < C$ uniformly in n . Also note that $V_n \geq 0$ and $\|V_n\|_{L^\infty(B)} < C$ for all n . So using theorem.3(sec III.2) of [1] we get v_n (up-to a subsequence) is bounded in $L_{loc}^\infty(B)$. Hence $\|\omega_n\|_{L^\infty(B_a(0))} < C$, which gives along with (2.31), $\|\omega_n\|_{L^\infty(B)} < C$, which is contradictory to our assumption.

Case II : Let $t_n \rightarrow 0$. Then from 2.22 we have

$$\lim_{n \rightarrow \infty} t_n \int_B e^{u_n} \phi_1 = - \lim_{n \rightarrow \infty} \int_B g(u_n) \phi_1. \quad (2.34)$$

Now from (2.31) we get for any $0 < a < 1$,

$$\lim_{n \rightarrow \infty} t_n \int_{B \setminus B_a(0)} e^{u_n} = 0. \quad (2.35)$$

Now choose a such that $\phi_1(a) = 1/2$. Then from (2.34) we have

$$t_n \int_{B_a(0)} e^{u_n} < \frac{t_n}{\phi_1(a)} \int_B e^{u_n} \phi_1 < 2 \int_B \phi_1 + o\left(\frac{1}{n}\right) < 3\pi. \quad (2.36)$$

Combining (2.35), (2.36) we have

$$\int_B V_n e^{v_n} = t_n \int_B e^{u_n} < 4\pi.$$

Using Cor.3(Sec III.1) of [1] we get $\|v_n\|_{L^\infty(B)} < C$. That is $\|\omega_n\|_{L^\infty(B)} < C$, for all n , which is contradictory to our assumption. \square

Proof of theorem(1.1):

Proof. In theorem(2.5) we have shown $\|u_t\|_{L^\infty}$ is uniformly bounded. Using regularity we get $\|u_t\|_{C^{1,\alpha}} < C$, for some positive constant C . Now take $\Omega \subset C_{rad}^{1,\alpha}$, where $\Omega := \{u : u(x) = u(|x|), u \in C^{1,\alpha}(B) \cap C^0(B), \|u\|_{C^{1,\alpha}(B)} < C\}$. Now take

$$S_t = I - \Delta^{-1} \{ \lambda_1 I + t(\exp \circ I + f) + (1-t)g \circ I \}.$$

Note that $0 \notin S(\partial\Omega)$ for all t . So using homotopy invariance we get $\deg(\Omega, S_0, 0) = \deg(\Omega, S_1, 0) = -1$. Hence the equation(1.2) has a radial solution. Also from the equation it is obvious that the solution is nontrivial for $f \neq -1$. \square

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